Pergamon

Topology Vol. 36, No. 6, pp. 1193–1220, 1997

0040-9383/97 $17.00 + 0.00

PII: S0040-9383(97)00004-9

APPLICATIONS OF CONTACT SURGERY

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(*Received* 13 *May* 1996; *in revised form* 29 *November* 1996)

1. INTRODUCTION

The first general result on the existence of contact structures is due to Martinet [22], who has shown that any closed orientable 3-manifold admits a contact structure (The basic notions of contact geometry are reviewed in Section 2 below). His proof is based on Lickorish's surgery description of 3-manifolds. Starting with the standard contact structure on S^3, one may assume the surgery curves to be transverse to the contact structure, and then surgery can be performed in such a way as to obtain a contact structure on the resulting manifold. In fact, the framing for each of these surgeries may be changed by a so-called Lutz twist, and this allows to prove that on a closed orientable 3-manifold there exists a contact structure in every homotopy class of almost contact structures (which here just means orientable 2-plane fields).

The connected sum theorem of Meckert [25] may be regarded as the first result in higher-dimensional contact topology. She proved that the connected sum of any two contact manifolds of dimension $2n + 1$ also admits a contact structure. By realizing certain indecomposable manifolds as Brieskorn varieties and applying Meckert's theorem, Thomas [30] was able to prove the existence of contact structures on a wide class of highly connected manifolds.

Martinet's surgery does not readily generalize to higher dimensions. In [1] Eliashberg realized that higher-dimensional contact surgery is indeed possible, provided the surgery spheres are taken to be *isotropic*, that is, tangent rather than transverse to the contact structure. Weinstein [34] has given a simplified description of part of this construction. We give a brief review of this contact surgery in Section 3.

In [6, 7] this was used to prove general existence results for contact structures first on simply-connected 5-manifolds and then, more generally, on highly connected manifolds of arbitrary (odd) dimension.

Due to an overinterpretation of Eliashberg's result, the proof of the main result in [7] is incomplete. In Section 4 of the present paper we set the record straight. Subject to a certain $\mathbb{Z}_2$-invariant being zero, it is shown that every $(n-1)$-connected $(2n+1)$-manifold admits a contact structure (possibly after taking the connected sum with a homotopy sphere), provided it admits an almost contact structure. It turns out that in several places a much deeper study of framing questions is required to be able to understand precisely which homotopy classes of almost contact structures can be realized by contact structures.

The remaining sections are devoted to further applications of contact surgery. In Section 5 we study exotic contact structures on spheres, that is, contact structures that are not diffeomorphic to the standard structure. The principal new result here is the construction of an exotic but homotopically standard contact structure on S^7 and S^{8k+3}, $k \geqslant 1$, and in principle on spheres of arbitrary odd dimension. For spheres of dimension $4n + 1$ we recover a theorem of Morita and Sato: Every homotopy class of almost contact structures on S^{4n+1} contains a contact structure.

In Section 6 we use an idea of Charles Thomas to find contact structures on certain quotients of S^5 under non-linear finite group actions by showing that a construction of Petrie carries over to contact topology. No such examples have been known previously.

In Section 7 we briefly discuss the notion of convex symplectic manifolds (due to Eliashberg–Gromov [4]) and indicate how contact surgery can be used to obtain such manifolds with non-trivial topology.

2. BASIC DEFINITIONS

A *contact manifold* $(M, \mathscr{D})$ is a smooth $(2n + 1)$-dimensional manifold M with a maximally non-integrable hyperplane distribution $\mathscr{D} \subset TM$. This means that locally one can find a 1-form ω defining $\mathscr{D} = \ker \omega$ and satisfying $\omega \wedge (d\omega)^n \neq 0$. Such a $\mathscr{D}$ is called a *contact structure*. If ω is globally defined, it is called a *contact form*. Necessary and sufficient for this is the coorientability of $\mathscr{D}$. In this case the structure group of TM reduces to $\mathrm{U}(n) \times 1$: A trivial line bundle complementary to $\mathscr{D}$ is defined by the *Reeb vector field* ξ, which is characterized by the equations

$$d\omega(\xi, .) \equiv 0 \quad \text{and} \quad \omega(\xi) \equiv 1;$$

on $\mathscr{D}$ we have a symplectic form $d\omega$ and a compatible complex bundle structure $J : \mathscr{D} \to \mathscr{D}$, that is, $d\omega(JX, JY) = d\omega(X, Y)$ for all $X, Y \in \mathscr{D}$ and $d\omega(X, JX) > 0$ for $X \neq 0$. Such a reduction of the structure group of the tangent bundle TM is called an *almost contact structure* and is written as $(\xi, \mathscr{D}, J)$. The almost contact structure compatible in this sense with a contact form ω is uniquely determined up to homotopy; in fact, it is determined by the contact structure since the conformal class of $d\omega | \mathscr{D}$ only depends on $\mathscr{D}$, not on the particular choice of ω.

The *standard contact structure* on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ is given by

$$\omega = \sum_{i=1}^{n+1} (x_i \, dy_i - y_i \, dx_i).$$

A contact structure on S^{2n+1} is called *homotopically standard* if its underlying almost contact structure extends as an almost complex structure over the disc D^{2n+2}. To be precise, an almost contact structure $(\xi, \mathscr{D}, J)$ on S^{2n+1} defines an almost complex structure $\tilde{J}$ on $TD^{2n+2} | S^{2n+1}$ if we require that $\tilde{J} | \mathscr{D} = J$ and $\tilde{J} v = \xi$, where v denotes a vector field along S^{2n+1} pointing outwards. If $\tilde{J}$ extends as an almost complex structure over D^{2n+2}, then $(\xi, \mathscr{D}, J)$ is homotopically standard. Of course, the standard contact structure is homotopically standard.

Now let (W, Ω) be a symplectic manifold with boundary $M = \partial W$. When restricted to TM the symplectic form Ω has a one-dimensional kernel. Suppose also that M admits a contact structure $\mathscr{D} = \ker \omega$ such that the kernel of Ω on TM is transverse to $\mathscr{D}$ and $\Omega | \mathscr{D}$ lies in the (positive) conformal class of $d\omega | \mathscr{D}$. Then (W, Ω) is called a *symplectic filling* of $(M, \mathscr{D})$.

The boundary $M = \partial W$ is called *globally* (resp. *locally*) *Ω-convex* if there exists an expanding Liouville vector field η on all W (resp. near M), that is, $L_\eta \Omega = d(\Omega(\eta, .)) = \Omega$, which is pointing outwards along M. Then (W, Ω) is a symplectic filling of the contact manifold $(M, \mathscr{D} = \ker \Omega(\eta, .))$. A locally Ω-convex boundary is also called a boundary of *contact type*. See [4] for more on these notions.

3. CONTACT SURGERY

Weinstein [34] gives a very readable account of the basic construction in contact surgery. If S^p is an isotropic sphere in a contact manifold (M^{2n+1}, ω) with trivial conformal symplectic normal bundle $CSN(M^{2n+1}, S^p) = (TS^p)^{\perp'}/TS^p$ (where $(TS^p)^{\perp'}$ denotes the symplectic orthogonal bundle with respect to the symplectic form $d\omega$ on the contact structure $\mathscr{D} = \ker \omega$; observe that $TS^p \subset (TS^p)^{\perp'} \subset \mathscr{D}$ for an isotropic sphere), then surgery can be performed along S^p, that is, we can cut out $S^p \times D^{2n+1-p}$ and glue back $D^{p+1} \times S^{2n-p}$. This glueing depends on the chosen trivialization of $CSN(M^{2n+1}, S^p)$, which induces a framing of S^p since

$$(TM^{2n+1}|S^p)/TS^p \cong (TM^{2n+1}|S^p)/\mathscr{D} \oplus \mathscr{D}/(TS^p)^{\perp'} \oplus (TS^p)^{\perp'}/TS^p$$
$$\cong \langle \xi \rangle \oplus T^*S^p \oplus CSN(M^{2n+1}, S^p)$$

and there is a natural trivialization of $\langle \xi \rangle \oplus T^*S^p$. Notice that if $p = n$, i.e. in the case of a *Legendre* sphere, the conformal symplectic normal bundle has rank zero and a priori there is no choice of framing at all. However, Eliashberg [1] has shown by a generating function argument that the framing can be changed by perturbing the Legendre sphere (through non-Legendre embeddings).

The usefulness of this result rests on the fact that there is an h-principle for isotropic spheres. The following is essentially Eliashberg's Proposition 2.3.1 (cf. also [6, Theorem 3]), but we fill in some details of the differential topological aspects of its proof.

LEMMA 1. *Let (M^{2n+1}, ω) be a contact manifold with contact structure $\mathscr{D} = \ker \omega$, $n \geqslant 2$. Let $i_0 : S^p \to M^{2n+1}$, $p \leqslant n$, be an embedding with trivial normal bundle. If i_0 is covered by a fibrewise injective complex bundle map $TS^p \otimes \mathbb{C} \to \mathscr{D}$, then there is an isotropic embedding $i_1 : S^p \to M^{2n+1}$, $i_1(TS^p) \subset \mathscr{D}$, which is isotopic to i_0. The embedding i_1 and the isotopy between i_0 and i_1 can be found C^0-close to i_0.*

Remark. The result remains true for $n = 1$, but the proof requires slightly different methods.

Proof. Regard i_0 as a map $i_0 : S^p \to S^p \times D^{2n+1-p} =: N$ into an arbitrarily small tubular neighbourhood of $i_0(S^p)$. By [13, p. 339] we find an isotropic immersion $i_1 : S^p \to N$ homotopic and C^0-close to i_0. After a generic contact perturbation (using contact Hamiltonians) we may assume that i_1 is an *embedding* homotopic to i_0. Since $2(2n + 1) \geqslant 3p + 4$ and $2p - (2n + 1) + 2 \leqslant p - 1$ for $p \leqslant n$ and $n \geqslant 2$ we are in the stable range where Haefliger's results apply (cf. [32]). These results say that homotopy classes of continuous maps $S^p \to N$ are in one-to-one correspondence with isotopy classes of embeddings. Hence, i_0 and i_1 are actually *isotopic* as embeddings (and the isotopy may be chosen C^0-close to i_0).

Next, the construction in Sections 2.3 and 2.4 of [1] (together with [34]) can be summarized as follows.

LEMMA 2 (Eliashberg [1], Weinstein [34]). *Let W be a symplectic manifold of dimension $2n + 2$, $n \geqslant 2$, and M a boundary component of contact type. Attach a $(p + 1)$-handle $(p \leqslant n)$ to W along M and call the resulting manifold W', and M' the new boundary component (which is the result of performing surgery on M along a p-sphere). If the compatible almost complex structure on W extends over the handle, then there is a symplectic structure on W' such that M' is of contact type. If $S^p \to M$ is an isotropic embedding such that $CSN(M, S^p)$ is trivial, such an extension exists for at least one particular choice of framing.*

Remark. (1) The situation for $n = 1$ is analysed in great detail in forthcoming work of Gompf [11].

(2) The rank of $CSN(M^{2n+1}, S^p)$ is $2(n-p)$. Hence, if $S^n \to M^{2n+1}$ is an embedding that satisfies the conditions of Lemma 1, one can always perform contact surgery along S^n for a suitable choice of framing.

As an immediate corollary we have the following theorem. Eliashberg actually proves a result about complex manifolds with strictly pseudoconvex boundary, but we keep the language of symplectic and contact geometry.

THEOREM 3 (Eliashberg [1]). *Let M^{2n+1} be the boundary of a handlebody W^{2n+2} $(n \geq 2)$ that contains only handles of index $\leq n+1$. Then M admits a contact structure in every homotopy class of almost contact structures that is induced from an almost complex structure on W.*

In [7] we claimed more than what Eliashberg has proved by erroneously omitting the last half-sentence in this theorem. This renders the proof of the main theorem of [7] incomplete. Although this hardly affects that theorem (in fact it may still be true as stated), the proof requires considerably more attention to details of Wall's classification of highly connected manifolds.

It is instructive to begin with the following example. Consider $M = S^2 \times S^3$. We may think of M as the boundary of a handlebody W with precisely one 3-handle and no other handles (in classical notation: $W \in \mathscr{H}(6,1,3)$). In other words, we regard M as an S^2-bundle over S^3, obtained from S^5 by surgery along S^2. It is easy to see from Weinstein's [34] description of contact surgery that this surgery (which is uniquely defined, since the choice of framing lies in the zero group $\pi_2(\mathrm{SO}_3)$) yields a contact structure on $S^2 \times S^3$ whose underlying almost contact structure has vanishing first Chern class c_1. Indeed, observe that $H_2(S^2 \times S^3; \mathbb{Z})$ is generated by the belt sphere S_b^2 of the 3-handle, and in contact surgery (along a Legendre sphere) this belt sphere is isotropic, i.e. again a Legendre sphere. Hence $T(S^2 \times S^3)|S_b^2 \cong (TS^2 \otimes \mathbb{C}) \oplus \varepsilon^1$, where ε^1 is a trivial line bundle transverse to the (cooriented) contact structure, and the first Chern class of a complexified real vector bundle is of order two, hence equal to zero in our situation.

The same is clear from Eliashberg's theorem. W is homotopy equivalent to S^3, hence every almost complex structure J on W has $c_1(W,J) = 0$, and the first Chern class of the almost contact structure on M is the pull-back of $c_1(W,J)$ under the inclusion $i: M \to W$.

On the other hand, we may think of $M = S^2 \times S^3$ as the boundary of a handlebody W' with precisely one 2-handle and no other handles, that is, $W' \in \mathscr{H}(6,1,2)$. This amounts to regarding M as an S^3-bundle over S^2, obtained from S^5 by surgery along S^1. As shown in [6] (see also Section 4.2 below), homotopy classes of almost contact structures on M are classified by c_1, and so are homotopy classes of almost complex structures on $W' \simeq S^2$. Since $i^*: H^2(W') \to H^2(M)$ is an isomorphism (for any coefficient group), application of Eliashberg's theorem yields a contact structure in every homotopy class of almost contact structures. An alternative argument, using Weinstein's description of contact surgery and the fact that now the framing may be changed by changing the (conformally symplectic) trivialization of the rank 2 bundle $CSN(S^5, S^1)$ is implicit in the proof of Theorem 8 of [6].

In conclusion, while it may not be possible to realize all (stable) homotopy classes of almost contact structures by contact structures starting from a fixed description of M as the boundary of a handlebody with handles of indices up to the middle dimension, it may be possible to do so using different such descriptions. We shall see that with a few minor exceptions, this approach allows to save the main theorem of [7].

4. HIGHLY CONNECTED MANIFOLDS

Let M be an $(n-1)$-connected $(2n+1)$-manifold, $n \geqslant 2$. We can split M as a connected sum $M = M_0 \# M_1 \# \cdots \# M_r$ with $H_n(M_0;\mathbb{Z})$ finite and $H_n(M_i;\mathbb{Z}) \cong \mathbb{Z}$ for $i = 1, \ldots, r$ (see [33, p. 285]). Of course we mean to include in this notation the case that $H_n(M;\mathbb{Z})$ is finite.

The following theorem, replacing Theorem 2 of [7], is not the best one can prove, but it is the easiest to state. Some other manifolds that also admit a contact structure are collected in Section 4.4. The invariant $\hat{\phi}(M_i)$ appearing in this theorem is an element of $H^{n+1}(M_i;\mathbb{Z}_2) \cong \mathbb{Z}_2$ $(i = 1, \ldots, r)$ for $n \neq 2, 6$ even and zero otherwise (see Section 4.1 below). The decomposition $M = M_0 \# M_1 \# \cdots \# M_k$ can always be chosen in such a way that $\hat{\phi}(M_i) = 0$ for $i = 1, \ldots, r-1$.

THEOREM 4. *Let $M = \#_{i=0}^r M_i$ be an $(n-1)$-connected $(2n+1)$-manifold (in the above notation). Assume that for $i = 1, \ldots, r$ we have $\hat{\phi}(M_i) = 0$.*

(1) *If M admits an almost contact structure, it is almost diffeomorphic to a manifold that admits a contact structure.*

(2a) *If n is odd, $n \not\equiv 3 \bmod 8$, then M is almost diffeomorphic to a manifold that admits a contact structure in every stable homotopy class of almost contact structures.*

(2b) *If $n \equiv 3 \bmod 8$, there is a top-dimensional $\mathbb{Z}_2$-obstruction to stable homotopy of almost contact structures. At least one of the two stable homotopy classes of almost contact structures (on a manifold almost diffeomorphic to M) corresponding to a fixed almost contact structure over the $(n+1)$-skeleton of M contains a contact structure.*

(2c) *If n is even, then M is almost diffeomorphic to a manifold that admits a contact structure in every homotopy class of almost contact structures.*

Remark. (1) *Almost diffeomorphism* means diffeomorphism up to the connected sum with a homotopy sphere.

(2) We only have to be concerned with realizing the stable homotopy class of almost contact structures; the argument that "stable" can be omitted for n even is as in [7]: On S^{2n+1}, n even, any homotopy class of almost contact structures is induced from a contact structure (cf. Section 5 below); by taking connected sums we can prove the same for any M.

Throughout this section M will denote an $(n-1)$-connected $(2n+1)$-manifold. When identifying two manifolds it is understood that such identification is only up to almost diffeomorphism. To establish our notation we briefly recall Wall's classification of highly connected manifolds.

4.1. Wall's classification

Strictly speaking, Wall [33] classifies *almost closed* manifolds, that is, manifolds bounded by a homotopy sphere. Furthermore, his classification contains certain exceptional cases that require additional invariants. We shall explain below (in somewhat more detail than in [7]) why we only have to be interested in the non-exceptional cases and why we may treat Wall's result as a classification of closed manifolds.

THEOREM 5 (Wall [33]). *Let M be an $(n-1)$-connected $(2n+1)$-manifold in the non-exceptional case, $n \geq 2$, $n \neq 3, 7$. Then M is, up to almost diffeomorphism, determined by*

A. homology invariants (essentially $H_n(M;\mathbb{Z})$ with its quadratic structure),
B. tangential invariants

(i) $\hat{\alpha} \in H^n(M;\pi_{n-1}(\mathrm{SO}))$,
(ii) $\hat{\beta} \in H^{n+1}(M;\pi_n(\mathrm{SO}))$,
(iii) *For $n \neq 2, 6$ even, $\hat{\phi} \in H^{n+1}(M;\mathbb{Z}_2) \cong H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_2$.*

Remark. The classification in the cases $n = 3$ and $n = 7$ was carried out by Wilkens [36]. For our purposes it will be sufficient to know that here $\hat{\alpha}$ is always zero.

We shall see below that if $n \equiv 0 \bmod 8$ a necessary condition for M to admit an almost contact structure is $\hat{\alpha}$ even, if $n \equiv 1 \bmod 8$ we need $\hat{\alpha} = 0$. Hence, by [33, p. 277], we are in the non-exceptional case. The only other exceptional case occurs for $n = 4$. That this case is indeed exceptional follows from [5], and then Theorem 9 of [33] states that the exceptional invariant $\omega \in \mathbb{Z}_2$ distinguishes between almost closed manifolds that can or cannot be closed with a disc. Since we are only interested in closed manifolds (the existence of contact structures on open manifolds being covered by Gromov's h-principle [14]), we may treat the case $n = 4$ as non-exceptional.

It then follows (again from Theorem 9 of [33]) that all the almost closed manifolds that admit almost contact structures can actually be closed, since the only obstruction to closing comes from the exceptional invariants and, in the case $n \equiv 0 \bmod 8$, from an $\hat{\alpha}$ which is not even.

4.2. Almost contact structures

The following is a slight modification of Proposition 4 from [7]. Throughout we assume $n \geq 2$.

PROPOSITION 6. *An $(n-1)$-connected $(2n+1)$-manifold M admits an almost contact structure in the following cases:*

(i) $n \equiv 0 \bmod 8$, *$\hat{\alpha}$ even, $\hat{\beta} = 0$, and $\hat{\phi} = 0$,*
(ii) $n \equiv 1 \bmod 8$ *and $\hat{\alpha} = 0$,*
(iii) $n \equiv 2 \bmod 8$ *and $\delta\hat{\alpha} = 0$,*
(iv) $n \equiv 3, 5, 6 \bmod 8$,
(v) $n \equiv 4 \bmod 8$ *and $\hat{\phi} = 0$,*
(vi) $n \equiv 7 \bmod 8$ *and $\hat{\beta}$ even.*

In each case the mentioned conditions are necessary, apart from $\hat{\phi} = 0$ in the cases $n \equiv 0, 4 \bmod 8$. The condition in (vi) *is automatically satisfied for $n = 7$.*

Remark. (1) Here δ denotes the Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

(2) The condition $\hat{\phi} = 0$ was left out in Proposition 4 of [7]. With the corrected statement of Eliashberg's theorem the proof given there is not complete. However, we shall see that the condition $\hat{\phi} = 0$ is indeed superfluous if $H_n(M;\mathbb{Z})$ is finite, and we conjecture that the same is true in general.

(3) For $n = 2$ the class $\hat{\alpha}$ can be identified as the second Stiefel–Whitney class w_2 (see [19]). Thus, together with Theorem 4 we recover the main theorem of [6]: A simply connected 5-manifold M admits a contact structure if and only if the third integral Stiefel–Whitney class $W_3 = \delta w_2$ vanishes, and M admits a contact structure in every homotopy class of almost contact structures.

Proof. For $n \not\equiv 0, 4 \bmod 8$ the argument outlined in [7] remains valid. We recall the parts of it that we shall need in the sequel and also include some additional details of the algebraic topology used in that argument, since these details are relevant to the proof of Lemma 7 below.

First of all we observed that M admits an almost contact structure if and only if it admits a stable almost complex structure, since all the coefficient groups of the relevant obstruction groups $H^q(M;\pi_{q-1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n))$, $1 \leqslant q \leqslant 2n+1$, are stable (even without any highly connectedness assumption on M). Therefore the existence of an almost contact structure is equivalent to the existence of a lifting of the stable tangent bundle $[TM] \in \widetilde{KO}(M)$ to $\widetilde{KU}(M)$. Denote the $(n+1)$-skeleton of M by $M^{(n+1)}$ and let x_0 be a point in $M^{(n+1)}$. The Atiyah–Hirzebruch spectral sequence in K-theory for the pair $(M^{(n+1)}, x_0)$ has E_2-term

$$E_2^{p,q} \cong H^p(M^{(n+1)}, x_0; K^q(*))$$

and converges to $K^*(M^{(n+1)}, x_0) = \tilde{K}^*(M^{(n+1)})$. Since $E_2^{p,q} \neq 0$ only for $p = n, n+1$, this spectral sequence collapses at the E_2-term, i.e. $E_\infty = E_2$. Now E_∞ is a filtration of $\tilde{K}^*$, so for the total degree 0 term $\tilde{K} = \tilde{K}^0$ we get

$$H^{n+1}(M;K^{-(n+1)}(*)) \rightarrowtail \tilde{K}(M^{(n+1)}) \twoheadrightarrow H^n(M;K^{-n}(*)).$$

With $KU^{-q}(*) = \pi_{q-1}(\mathrm{U})$ and $KO^{-q}(*) = \pi_{q-1}(\mathrm{SO})$ for $q \geqslant 2$ this yields the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H^{n+1}(M;\pi_n(\mathrm{U})) & \rightarrowtail & \widetilde{KU}(M^{(n+1)}) & \twoheadrightarrow & H^n(M;\pi_{n-1}(\mathrm{U})) \\ \downarrow & & \downarrow & & \downarrow \\ H^{n+1}(M;\pi_n(\mathrm{SO})) & \rightarrowtail & \widetilde{KO}(M^{(n+1)}) & \twoheadrightarrow & H^n(M;\pi_{n-1}(\mathrm{SO})). \end{array}$$

Here the coefficient groups are as follows:

$n \bmod 8$	$\pi_n(\mathrm{U})$	$\pi_n(\mathrm{SO})$	$\pi_n(\mathrm{U}) \to \pi_n(\mathrm{SO})$
0	0	$\mathbb{Z}_2$	
1	$\mathbb{Z}$	$\mathbb{Z}_2$	mod 2 reduction
2	0	0	
3	$\mathbb{Z}$	$\mathbb{Z}$	identity
4	0	0	
5	$\mathbb{Z}$	0	
6	0	0	
7	$\mathbb{Z}$	$\mathbb{Z}$	multiplication by 2

Remark. Notice that from this table of coefficients it follows that the short exact sequences in the commutative diagram above are split except for $n \equiv 1 \bmod 8$. However, here we need $\hat{\alpha} = 0$ for a lift to the zero group $H^n(M;\pi_{n-1}(\mathrm{U}))$ to exist, which is a necessary condition for the existence of an almost contact structure on M. Then again, as in all the other cases, we can indeed define an invariant $\hat{\beta} = \hat{\beta}(M) \in H^{n+1}(M;\pi_n(\mathrm{SO}))$.

If one considers the Atiyah–Hirzebruch spectral sequence in K-theory for the pair (M, x_0) instead of $(M^{(n+1)}, x_0)$ one has to take care of the differentials

$$d_n^{n+1,-n-1}: H^{n+1}(M;\pi_n(G)) \to H^{2n+1}(M;\pi_{2n-1}(G))$$

and

$$d_{n+1}^{n,-n}: H^n(M;\pi_{n-1}(G)) \to H^{2n+1}(M;\pi_{2n-1}(G)),$$

which may now be non-zero ($G = \mathrm{U}, \mathrm{SO}$, respectively).

The $d_r^{p,q} = d_r^{p,q}(M)$ are cohomology operations, hence by a well-known result of Serre (cf. [35, V.8]) we may regard $d_r^{p,q}$ as element of $H^{p+r}(K(A,p);B)$, where $A = K^q(*)$, $B = K^{q-r+1}(*)$, and $K(A,p)$ is the Eilenberg–MacLane space of type (A,p).

In the complex case, since $\pi_k(\mathrm{U}) = 0$ for k even, we only have to consider

$$d_n^{n+1,-n-1} \in H^{2n+1}(K(\mathbb{Z}, n+1);\mathbb{Z}) \text{ for } n \text{ odd}$$

and

$$d_{n+1}^{n,-n} \in H^{2n+1}(K(\mathbb{Z}, n);\mathbb{Z}) \text{ for } n \text{ even}.$$

Both these cohomology groups are finite [35, p. 670], so the cohomology operation $d_n^{n+1,-n-1}(M)$ (resp. $d_{n+1}^{n,-n}(M)$) has to vanish since it maps into the infinite cyclic group $H^{2n+1}(M;\mathbb{Z})$.

So the spectral sequence for (M, x_0) in KU-theory also collapses at the E_2-term. Since $H^{2n+1}(M;\pi_{2n}(\mathrm{U})) = 0$ there are no new entries of total degree 0 in the E_2-page, so we have the same short exact sequence for $\widetilde{KU}(M)$ as for $\widetilde{KU}(M^{(n+1)})$. An easy application of the Five-Lemma shows that the isomorphism $\widetilde{KU}(M) \cong \widetilde{KU}(M^{(n+1)})$ is induced by the inclusion $M^{(n+1)} \to M$.

Consider now the following piece of the long exact sequence in KO-theory for the pair $(M, M^{(n+1)})$, where Σ denotes suspension.

$$\widetilde{KO}(\Sigma M^{(n+1)}) \to KO(M, M^{(n+1)}) \to \widetilde{KO}(M) \to \widetilde{KO}(M^{(n+1)}).$$

Since M is obtained from $M^{(n+1)}$ by attaching a $(2n+1)$-cell, we have

$$KO(M, M^{(n+1)}) = \widetilde{KO}(M/M^{(n+1)}) = \widetilde{KO}(S^{2n+1}).$$

For $n \not\equiv 0, 4 \bmod 8$ we have $\widetilde{KO}(S^{2n+1}) = 0$, so we get the commutative diagram

$$\begin{tikzcd} & \widetilde{KU}(M) \arrow[r, "\cong"] \arrow[d] & \widetilde{KU}(M^{(n+1)}) \arrow[d] \\ 0 \arrow[r] & \widetilde{KO}(M) \arrow[r] & \widetilde{KO}(M^{(n+1)}). \end{tikzcd}$$

This shows that there are no obstructions to the existence of an almost contact structure in $H^{2n+1}(M;\pi_{2n}(\mathrm{SO}/\mathrm{U}))$, and the proposition follows from a simple case by case study of the first commutative diagram above.

For $n \equiv 0, 4 \bmod 8$ we shall see below that under the stated assumptions M is almost diffeomorphic to a manifold that admits a contact structure. Then the result follows from Proposition 23 below, which states that a manifold that is almost diffeomorphic to an almost contact manifold does itself admit an almost contact structure.

The same argument can be applied to a $(2n+2)$-dimensional handlebody with n-handles only, $W \in \mathscr{H}(2n+2, k, n)$, which is homotopy equivalent to a wedge of n-spheres, $W \simeq \bigvee_k S^n$, or a handlebody $W \in \mathscr{H}(2n+2, k, n+1) \simeq \bigvee_k S^{n+1}$ (as well as to some other handlebodies that are not relevant to our discussion below). These spaces are $(n-1)$-

connected and have no homology above dimension $n+1$. So the following is immediate from the arguments used to prove the preceding proposition.

LEMMA 7. (i) *The (stable) tangent bundle of the handlebody* $W \in \mathscr{H}(2n+2,k,n)$ *is characterized by an element* $\hat{\alpha}_W \in H^n(W;\pi_{n-1}(\mathrm{SO}))$ *and* W *admits an almost complex structure if and only if* $\hat{\alpha}_W$ *lifts to an element* $\tilde{\alpha}_W \in H^n(W;\pi_{n-1}(\mathrm{U}))$.

(ii) *The (stable) tangent bundle of the handlebody* $W \in \mathscr{H}(2n+2,k,n+1)$ *is completely determined by an element* $\hat{\beta}_W \in H^{n+1}(W;\pi_n(\mathrm{SO}))$; *the necessary and sufficient condition for* W *to admit an almost complex structure is the existence of a lifting of* $\hat{\beta}_W$ *to* $\tilde{\beta}_W \in H^{n+1}(W;\pi_n(\mathrm{U}))$.

Remark. The conditions for these lifts to exist are as in Proposition 6.

The lifts $\tilde{\alpha}_W$ and $\tilde{\beta}_W$ (if they exist) determine a unique homotopy class of almost complex structures over W. Similarly, if $\hat{\alpha}, \hat{\beta}$ satisfy the conditions of Proposition 6, there exists lifts $\tilde{\alpha} \in H^n(M;\pi_{n-1}(\mathrm{U}))$ and $\tilde{\beta} \in H^{n+1}(M;\pi_n(\mathrm{U}))$ which determine a unique homotopy class of almost contact structures over the $(n+1)$-skeleton of M. If this almost contact structure extends over M (which may depend on $\hat{\phi}$ being zero in the cases $n \equiv 0,4 \bmod 8$), the obstruction to two such extensions being stably homotopic lies in

$$H^{2n+1}(M;\pi_{2n+1}(\mathrm{SO/U})) \cong \pi_{2n+1}(\mathrm{SO/U}) \cong \begin{cases} \mathbb{Z}_2 & \text{for } n \equiv 3 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

This proves the following.

LEMMA 8. *Assume that the conditions of Proposition* 6 *are satisfied. Then the lifts* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *of* $\hat{\alpha}$ *and* $\hat{\beta}$, *respectively, determine a unique stable homotopy class of almost contact structures on M unless* $n \equiv 3 \bmod 8$, *in which case there are exactly two different extensions (up to stable homotopy) to an almost contact structure on M.*

If we are interested in the classification of almost contact structures up to homotopy rather than stable homotopy, the relevant coefficient groups are the first non-stable homotopy groups $\pi_{2n+1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n)$, which have been determined by Massey [23]:

$$\pi_{2n+1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n) \cong \begin{cases} \mathbb{Z}_{n!} & \text{for } n \equiv 0 \bmod 4 \\ \mathbb{Z} & \text{for } n \equiv 1 \bmod 4 \\ \mathbb{Z}_{n!/2} & \text{for } n \equiv 2 \bmod 4 \\ \mathbb{Z} + \mathbb{Z}_2 & \text{for } n \equiv 3 \bmod 4. \end{cases}$$

Using work of Morita [27] that will also be important in Section 5 below, Sato [29] has shown that S^{2n+1} admits a contact structure in every homotopy class of almost contact structures if n is even (in which case there are only finitely many such homotopy classes), and contact structures in infinitely many different homotopy classes of almost contact structures if n is odd. (However, as we shall see in Section 5, in the latter case Sato's argument is not strong enough to produce a contact structure in every homotopy class of almost contact structures.) Therefore we can reduce the proof of Theorem 4 to showing the existence of a contact structure in every stable homotopy class of almost contact structures (or at least those for which a certain $\mathbb{Z}_2$-invariant vanishes if $n \equiv 3 \bmod 8$), for the homotopy class of the underlying almost contact structure may be changed in the top dimension by taking the connected sum with a standard sphere with homotopically non-standard contact structure.

4.3. Proof of Theorem 4

By Meckert's connected sum theorem we may restrict our attention to indecomposable manifolds.

(A) Consider the case $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\hat{\phi}(M) = 0$. First we collect some topological information. This has already been used in [7] and can easily be extracted from Wall's series of papers on the classification of highly connected manifolds, but it is included here for the reader's convenience.

LEMMA 9. *If $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\hat{\beta}(M) = 0$, $\hat{\phi}(M) = 0$, then M is an S^{n+1}-bundle over S^n defined by $\hat{\alpha}(M) \in H^n(M;\pi_{n-1}(\mathrm{SO})) \cong \pi_{n-1}(\mathrm{SO})$.*

Proof. Let M' be the S^{n+1}-bundle over S^n defined by $\alpha_0 = \hat{\alpha}(M) \in \pi_{n-1}(\mathrm{SO}) \cong \pi_{n-1}(\mathrm{SO}_{n+2})$. It follows from the homotopy exact sequence and the Gysin sequence of this bundle that M' is $(n-1)$-connected with $H_n(M';\mathbb{Z}) \cong \mathbb{Z}$. Let S^n be an embedded sphere in M' representing the generator of $H_n(M';\mathbb{Z})$; such a sphere exists by the Hurewicz theorem and since we are in the "stable range" as defined in [32, p. 254].

Consider the long exact homology sequence of the pair (W', M'), where W' is the associated D^{n+2}-bundle over S^n:

$$\cdots \to H_{n+1}(W', M') \to H_n(M') \to H_n(W') \to H_n(W', M') \to \cdots$$

By Poincaré duality, $H_n(W', M') \cong H^{n+2}(W') = 0$, and similarly we have $H_{n+1}(W', M') = 0$. Hence $i(S^n)$ represents the generator of $H_n(W', \mathbb{Z})$, where $i\colon M' \to W'$ is the natural inclusion, and $i_*\colon H_n(M') \to H_n(W')$ is an isomorphism for any (finitely generated abelian) coefficient group.

The normal bundle $\nu(S^n, M')$ of S^n in M' is classified by an element $\hat{\alpha}(M') \in \pi_{n-1}(\mathrm{SO}_{n+1})$. The normal bundle $\nu(i(S^n), W')$ of $i(S^n)$ in W' is classified by $\alpha_0 \in \pi_{n-1}(\mathrm{SO}_{n+2})$, because $i(S^n)$ is isotopic to the zero section of W' (again we are in the stable range). Since $\nu(i(S^n), W') \cong \nu(S^n, M') \oplus \varepsilon^1$, where ε^1 denotes a trivial line bundle, and $\pi_{n-1}(\mathrm{SO}_{n+1})$ is stable, we have $\hat{\alpha}(M') = \alpha_0 = \hat{\alpha}(M)$.

The fibre S^{n+1} of M' represents the generator of $H_{n+1}(M', \mathbb{Z})$, as can be seen from the Wang sequence. Since it has trivial normal bundle, it follows that $\hat{\beta}(M') = 0$ and $\hat{\phi}(M') = 0$.

Thus M and M' have the same invariants, which implies that they are almost diffeomorphic.

So in the situation of Lemma 9 we have $M = \partial W$ with $W \in \mathscr{H}(2n+2, 1, n)$, and the natural homomorphism $i^*\colon H^n(W) \to H^n(M)$ is an isomorphism for any coefficient group.

LEMMA 10. *If $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\hat{\alpha}(M) = 0$, then M is an S^n-bundle over S^{n+1} with Euler class zero. Thus $M = \partial W$ with $W \in \mathscr{H}(2n+2, 1, n+1)$. Furthermore, the natural homomorphism $i^*\colon H^{n+1}(W) \to H^{n+1}(M)$ is an isomorphism for any coefficient group.*

Proof. Represent the generator of $H_n(M;\mathbb{Z})$ be an embedded S^n. The fact that $\hat{\alpha}(M) = 0$ implies that S^n has trivial normal bundle. Performing surgery along S^n has the effect of killing $H_n(M;\mathbb{Z})$, no matter what framing we choose (see Lemma 1 in [31]). The complementary surgery is again along an S^n, that is, M is obtained from a (homotopy) $(2n+1)$-sphere by surgery along S^n. Any embedding of S^n in S^{2n+1} is isotopic to the trivial embedding, and hence M is as described in the lemma.

It is immediate from the Gysin sequence for an S^n-bundle over S^{n+1} that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ precisely if the Euler class is zero.

Finally, the long exact homology sequence with integer coefficients of the pair (W, M) becomes

$$H_{n+2}(W,M) \to H_{n+1}(M) \xrightarrow{i_*} H_{n+1}(W) \to H_{n+1}(W,M) \xrightarrow{\Delta} H_n(M) \to H_n(W)$$

and with Poincaré duality $H_{n+2}(W,M) \cong H^n(W) = 0$ and $H_{n+1}(W,M) \cong H^{n+1}(W) \cong \mathbb{Z}$, hence

$$0 \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z} \to 0.$$

This implies that Δ is surjective and hence an isomorphism, and thus i_* is an isomorphism.

Since $\operatorname{Ext}(H_n(M;\mathbb{Z}), G) = \operatorname{Ext}(\mathbb{Z}, G) = 0$ for any coefficient group G and $\operatorname{Ext}(H_n(W;\mathbb{Z}), G) = \operatorname{Ext}(0, G) = 0$, we find that the homomorphism $i^*: H^{n+1}(W; G) \to H^{n+1}(M; G)$ is an isomorphism for any G by the universal coefficient theorem.

Remark. In the situation of Lemma 10, the S^n-bundle is determined by the invariant $\hat{\beta}$ and $\hat{\phi}$ as follows.

If $n = 3, 7$, the suspension map

$$S: S\pi_n(\mathrm{SO}_n) \to \pi_n(\mathrm{SO})$$

is injective with image of index 2, and the S^n-bundle over S^{n+1} is defined by $\beta_0 \in \pi_n(\mathrm{SO}_{n+1})$ with $S^{-1}(\hat{\beta}) = \beta_0 \in S\pi_n(\mathrm{SO}_n) \subset \pi_n(\mathrm{SO}_{n+1})$.

If $n \neq 3, 7$ is odd or $n = 2, 6$, the suspension map

$$S: S\pi_n(\mathrm{SO}_n) \to \pi_n(\mathrm{SO})$$

is bijective, and we want $\beta_0 = S^{-1}(\hat{\beta})$.

If $n \neq 2, 6$ is even, we have the split exact sequence

$$0 \to \mathbb{Z}_2 \to S\pi_n(\mathrm{SO}_n) \xrightarrow{S} \pi_n(\mathrm{SO}) \to 0.$$

Denote a splitting by S^{-1}; then we want $\beta_0 = (\hat{\phi}, S^{-1}(\hat{\beta}))$.

LEMMA 11. *The unit cotangent bundle M of S^{n+1} for $n \neq 2, 6$ even generates the kernel of the suspension homomorphism S, that is, it is characterized by $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$, $\hat{\alpha} = 0$, $\hat{\beta} = 0$, and $\hat{\phi} \neq 0$.*

Proof. The homotopy exact sequence shows that M is $(n-1)$-connected. The bundle has zero Euler class, so the Gysin sequence yields $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. The fibre S^n represents the generator of $H_n(M;\mathbb{Z})$ and has trivial normal bundle, hence $\hat{\alpha} = 0$.

Now we have $\pi_{n+1}(M) \cong H_{n+1}(M;\mathbb{Z}) \oplus (H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_2)$, where the surjection onto the first factor is given by the Hurewicz homomorphism and the injection of the second factor is given by composing elements of $H_n(M;\mathbb{Z}) \cong \pi_n(M)$ with the generator of $\pi_{n+1}(S^n)$ (note that M can be replaced by a homotopy equivalent CW-complex with cells of dimension $0, n, n+1$ and $2n+1$ only); cf. [33, p. 276]. By Proposition 1 of [32], any element of $\pi_{n+1}(M)$, in particular a generator of $H_{n+1}(M;\mathbb{Z})$, can be represented by an embedded sphere S_1^{n+1} (since $n \geqslant 4$). By Lemma 10, $i_*: H_{n+1}(M) \to H_{n+1}(W)$ is an isomorphism, where W is the cotangent disc bundle of S^{n+1}. Hence $i(S_1^{n+1})$ represents a generator of $H_{n+1}(W;\mathbb{Z})$. Another generator is given by the zero section S_0^{n+1} of W, and by Lemma 1 of [32], $i(S_1^{n+1})$ and S_0^{n+1} are regularly homotopic (again this lemma applies for $n \geqslant 4$). Since the cotangent sphere bundle of S^{n+1} is stably trivial, we find $\hat{\beta}_W = 0$, and then also $\hat{\beta} = 0$ because of $\nu(S_1^{n+1}, M) \oplus \varepsilon^1 \cong \nu(i(S_1^{n+1}), W)$. (A shorter argument for $\hat{\beta} = 0$ can be given by interpreting it as obstruction class and using naturality.)

On the other hand, for $n \neq 2, 6$ even the cotangent bundle of S^{n+1} is not trivial, and hence it is detected by $\hat{\phi} \neq 0$.

Now we return to the general situation $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\hat{\phi}(M) = 0$ and deal with the different values of $n \bmod 8$ in turn.

(i) $n \equiv 0 \bmod 8$. Necessary conditions for M to admit an almost contact structure are $\hat{\alpha}$ even and $\hat{\beta} = 0$. Thus we are in the situation of Lemma 9. The lift $\tilde{\alpha}$ of $\hat{\alpha}$ to $H^n(M;\pi_{n-1}(\mathrm{U}))$ is unique, and $\hat{\beta}$ also lifts uniquely to $\tilde{\beta} = 0 \in H^{n+1}(M;\pi_n(\mathrm{U}))$ (which in fact is the zero group). These lifts determine the homotopy class of an almost contact structure over the $(n+1)$-skeleton of M.

To prove that there is an almost contact structure on M in the stable homotopy class determined by $\tilde{\alpha}$ and $\tilde{\beta}$ (remember that this part of Proposition 6 still has to be proved), we consider the handlebody $W \in \mathscr{H}(2n+2, 1, n)$ with $\partial W = M$. Let $\hat{\alpha}_W$ and $\hat{\beta}_W$ be the invariants that characterize the tangent bundle of W. (Since $W \simeq S^n$ has no cohomology in dimension $n+1$ we have of course $\hat{\beta}_W = 0$, but we keep the general notation because the argument in the other cases will be similar.) Then $i^*\hat{\alpha}_W = \hat{\alpha}$ and $i^*\hat{\beta}_W = \hat{\beta}$ since $TW \mid M = TM \oplus \varepsilon^1$.

By the remark after the proof of Lemma 9 the class $\hat{\alpha}_W$ is uniquely determined by the condition $i^*\hat{\alpha}_W = \hat{\alpha}$, and the unique $\tilde{\alpha}_W$ with $i^*\tilde{\alpha}_W = \tilde{\alpha}$ is a lift of $\hat{\alpha}_W$. Both $\hat{\beta}_W$ and $\tilde{\beta}_W$ lie in the zero group and we have $i^*\tilde{\beta}_W = \tilde{\beta} = 0$.

Thus there is an almost complex structure on W corresponding to these lifts $\tilde{\alpha}_W$, $\tilde{\beta}_W$. This almost complex structure induces an almost contact structure on $M = \partial W$ (which proves this particular case of Proposition 6) and, by Eliashberg's theorem, we can actually find a contact structure on M in this homotopy class of almost contact structures.

(ii) $n \equiv 1 \bmod 8$. Now the condition for M to admit an almost contact structure is $\hat{\alpha} = 0$, so we are in the situation of Lemma 10, and we take $W \in \mathscr{H}(2n+2, 1, n+1)$ as described there.

The lift of $\hat{\alpha} = 0$ to $\tilde{\alpha} = 0$ is unique, and $\hat{\alpha}_W$ as well as $\tilde{\alpha}_W$ lie in the zero group since $W \simeq S^{n+1}$.

Since $i^*: H^{n+1}(W) \to H^{n+1}(M)$ is an isomorphism for any coefficient group, any lift of the $\mathbb{Z}_2$-class $\hat{\beta} \in H^{n+1}(M;\pi_n(\mathrm{SO}))$ to an integral class $\tilde{\beta} \in H^{n+1}(M;\pi_n(\mathrm{U}))$ defines a lift $(i^*)^{-1}(\tilde{\beta})$ of $(i^*)^{-1}(\hat{\beta})$.

(iii) $n \equiv 2, 4, 6 \bmod 8$. Here $\hat{\beta}$ and its lift lie in the zero group. Hence the argument is as in (i). That same argument also concludes the proof of Proposition 6 in the case $n \equiv 4 \bmod 8$ for the particular M considered here.

(iv) $n \equiv 3, 5, 7 \bmod 8$. Here $\hat{\alpha}$ and its lift $\tilde{\alpha}$ lie in the zero group. Now argue as in (ii). Note, however, that for $n \equiv 3 \bmod 8$ there are two stable homotopy classes of almost contact structures corresponding to a fixed choice of $\tilde{\beta}$, and we can only guarantee to realize one of them.

(B) Next we consider the case $H_n(M;\mathbb{Z})$ finite. We observe that in this case either $\hat{\alpha}(M)$ is zero because it lies in the zero group $H^n(M;\mathbb{Z})$, or — when the coefficient group is $\pi_{n-1}(\mathrm{SO}) = \mathbb{Z}_2$ — it has to be zero for an almost contact structure to exist. For in all cases $\tilde{\alpha} = 0$ is the only possible lift to $H^n(M;\pi_{n-1}(\mathrm{U}))$ (which is always the zero group).

By [33, Theorem 8] the fact that $\hat{\alpha} = 0$ implies the existence of $W \in \mathscr{H}(2n+2, k, n+1)$ with $\partial W = M$. Thus W is homotopy equivalent to a wedge of k copies of S^{n+1}, and using Poincaré duality as before the long exact homology sequence of the pair (W, M) becomes

$$\begin{array}{ccccccc} H^n(M;G) & \overset{\Delta}{\rightarrowtail} & H^{n+1}(W,M;G) & \xrightarrow{j^*} & H^{n+1}(W;G) & \xrightarrow{i^*} & H^{n+1}(M;G) \\ H^n(M;G) & \rightarrowtail & G^{\oplus k} & \longrightarrow & G^{\oplus k} & \twoheadrightarrow & H^{n+1}(M;G). \end{array}$$

Now both $\hat{\alpha}_W$ and $\tilde{\alpha}_W$ lie in the zero group, so we only have to be concerned with finding a suitable lift $\tilde{\beta}_W \in H^{n+1}(W;\pi_n(\mathrm{U}))$ of $\hat{\beta}_W \in H^{n+1}(W;\pi_n(\mathrm{SO}))$. To study this question, we consider the commutative diagram

$$\begin{array}{ccccccc} H^n(M;G) & \overset{\Delta}{\rightarrowtail} & H^{n+1}(W,M;G) & \xrightarrow{j^*} & H^{n+1}(W;G) & \overset{i^*}{\twoheadrightarrow} & H^{n+1}(M;G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(M;G') & \overset{\Delta}{\rightarrowtail} & H^{n+1}(W,M;G') & \xrightarrow{j^*} & H^{n+1}(W;G') & \overset{i^*}{\twoheadrightarrow} & H^{n+1}(M;G') \end{array}$$

where $G = \pi_n(\mathrm{U})$ and $G' = \pi_n(\mathrm{SO})$. The vertical maps are induced from the coefficient map $\pi_n(\mathrm{U}) \to \pi_n(\mathrm{SO})$ and will be denoted by π.

(i) $n \equiv 0 \bmod 8$. Here the commutative diagram becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(M;\mathbb{Z}_2) & \overset{\Delta}{\rightarrowtail} & \mathbb{Z}_2^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}_2^{\oplus k} & \overset{i^*}{\twoheadrightarrow} & H^{n+1}(M;\mathbb{Z}_2). \end{array}$$

A necessary condition for an almost contact structure to exist is $\hat{\beta} = 0$. To show that there is a contact structure on M and, a fortiori, that there is no obstruction to an almost contact structure in $H^{2n+1}(M;\pi_{2n}(\mathrm{SO}/\mathrm{U}))$ we need to have an almost complex structure on W. For this we need $\hat{\beta}_W = 0$, which is not guaranteed a priori.

Now we observe that M is stably parallelizable since $\hat{\alpha} = 0$ and $\hat{\beta} = 0$ and there is no obstruction to stable parallelizability in $H^{2n+1}(M;\pi_{2n}(\mathrm{SO}))$. The latter fact follows from the same argument used to prove Theorem 3.1 (case 3) in [20]. (This argument, by the way, works in the other dimensions as well, so that quite generally $\hat{\alpha}$ and $\hat{\beta}$ are the only obstructions to stable parallelizability of a highly connected manifold.)

But then by Theorem 6.6 of [20] we may actually assume that the handlebody W is also stably parallelizable since we may kill $\pi_n(M)$ by *framed* surgery. In other words, we can ensure that $\hat{\beta}_W = 0$. Then we have an almost complex structure on W corresponding to the lift $\tilde{\beta}_W = 0$, this in turn induces an almost contact structure on $M = \partial W$ in the stable homotopy class determined by $\tilde{\beta} = 0$ (which concludes the proof of Proposition 6 for $n \equiv 0 \bmod 8$, and we see that the condition $\hat{\phi} = 0$ is redundant if $H_n(M;\mathbb{Z})$ is finite). By Eliashberg's theorem this almost contact structure is induced from a contact structure.

(ii) $n \equiv 1 \bmod 8$. Here we have the following commutative diagram, where the vertical maps are induced by mod 2 reduction $\mathbb{Z} \to \mathbb{Z}_2$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \overset{i^*}{\twoheadrightarrow} & H^{n+1}(M;\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(M;\mathbb{Z}_2) & \overset{\Delta}{\rightarrowtail} & \mathbb{Z}_2^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}_2^{\oplus k} & \overset{i^*}{\twoheadrightarrow} & H^{n+1}(M;\mathbb{Z}_2). \end{array}$$

The result in Theorem 4 follows in this case from the following lemma.

LEMMA 12. *Any lift $\tilde{\beta} \in H^{n+1}(M;\mathbb{Z})$ of $\hat{\beta} \in H^{n+1}(M;\mathbb{Z}_2)$ is of the form $i^*\tilde{\beta}_W$, where $\tilde{\beta}_W$ is a suitable lift of $\hat{\beta}_W$.*

Proof. Choose $\tilde{\gamma}_W \in H^{n+1}(W;\mathbb{Z})$ with $i^*\tilde{\gamma}_W = \tilde{\beta}$. Then $\hat{\beta} = \pi\tilde{\beta} = \pi i^*\tilde{\gamma}_W = i^*\pi\tilde{\gamma}_W$. On the other hand, $\hat{\beta} = i^*\hat{\beta}_W$. Hence $\hat{\beta}_W - \pi\tilde{\gamma}_W \in \ker i^* = \operatorname{im} j^*$. Write $\hat{\beta}_W - \pi\tilde{\gamma}_W = j^*\eta$ with $\eta \in H^{n+1}(W,M;\mathbb{Z}_2)$. Let $\tilde{\eta} \in H^{n+1}(W,M;\mathbb{Z})$ be a lift of η. Then $\hat{\beta}_W - \pi\tilde{\gamma}_W = j^*\pi\tilde{\eta} = \pi j^*\tilde{\eta}$, hence $\hat{\beta}_W = \pi(\tilde{\gamma}_W + j^*\tilde{\eta})$. Set $\tilde{\beta}_W = \tilde{\gamma}_W + j^*\tilde{\eta}$. Then $i^*\tilde{\beta}_W = i^*\tilde{\gamma}_W = \tilde{\beta}$.

(iii) $n \equiv 2, 4, 6 \bmod 8$. Here $\hat{\beta}$ and $\tilde{\beta}$ lie in the zero group, and so do $\hat{\beta}_W$ and $\tilde{\beta}_W$, since the two relevant coefficient groups are zero. Again we also get for free the vanishing of the top-dimensional obstruction to an almost contact structure in the case $n \equiv 4 \bmod 8$ (even if $\hat{\phi} \neq 0$). Thus the proof of Proposition 6 is now complete.

Remark. Strictly speaking, we have proved Proposition 6 (for $n \equiv 0, 4 \bmod 8$) only for manifolds with $H_n(M;\mathbb{Z})$ isomorphic to $\mathbb{Z}$ or finite. We shall see in Section 5 below that the connected sum of two almost contact manifolds does also admit an almost contact structure.

(iv) $n \equiv 3 \bmod 8$. Here the commutative diagram becomes

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \xrightarrow{i^*} & H^{n+1}(M;\mathbb{Z}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \xrightarrow{i^*} & H^{n+1}(M;\mathbb{Z}). \end{array}$$

All the vertical maps π are the identity map, hence the unique lift $\tilde{\beta}_W$ of $\hat{\beta}_W$ satisfies $i^*\tilde{\beta}_W = \tilde{\beta}$, where $\tilde{\beta}$ is the unique lift of $\hat{\beta}$.

(v) $n \equiv 5 \bmod 8$. Now we have the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \xrightarrow{i^*} & H^{n+1}(M;\mathbb{Z}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Let $\tilde{\beta}$ be a lift of $\hat{\beta} = 0$. Since i^* is surjective, there is a $\tilde{\beta}_W$ with $i^*\tilde{\beta}_W = \tilde{\beta}$, and trivially $\pi\tilde{\beta}_W = 0 = \hat{\beta}_W$.

(vi) $n \equiv 7 \bmod 8$. Here we have

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \xrightarrow{i^*} & H^{n+1}(M;\mathbb{Z}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & \mathbb{Z}^{\oplus k} & \xrightarrow{j^*} & \mathbb{Z}^{\oplus k} & \xrightarrow{i^*} & H^{n+1}(M;\mathbb{Z}). \end{array}$$

Now the vertical maps are multiplication by two, so the condition for M to admit an almost contact structure is $\hat{\beta}$ even. For W to admit an almost complex structure we need $\hat{\beta}_W$ even, and a priori it is not clear that W can be chosen in such a way. As in the case $n \equiv 0 \bmod 8$ we need to work a little harder to see that this can indeed be achieved.

We start with $W_0 = M \times [0,1]$, where we assume that M admits an almost contact structure, i.e. $\hat{\beta}$ even. Then W_0 admits a natural almost complex structure. We now want to attach $(n+1)$-handles to W_0 along $M \times \{1\}$ to kill $H_n(M;\mathbb{Z})$. First we consider an arbitrary n-sphere embedded in $M = M \times \{1\}$. Since $\hat{\alpha} = 0$, this n-sphere has trivial normal bundle in M. There is an almost contact structure on M, so in particular on an open tubular neighbourhood $S^n \times U^{n+1}$ of our n-sphere (Here U^{n+1} denotes the open unit disc). By the h-principle for contact structures on open manifolds [14] we can find an auxiliary contact structure on $S^n \times U^{n+1}$ which induces the given almost contact structure. By Eliashberg's theorem we can perform contact surgery along S^n (inside the neighbourhood $S^n \times U^{n+1}$). In particular, we can attach an almost complex $(n+1)$-handle to $M \times [0,1]$ along S^n.

To ensure that this surgery reduces the order of $H_n(M;\mathbb{Z})$ we have to choose a particular framing. By the proof of Lemma 2.4.1 of [1] our choice of framing is given by the kernel of $\pi_n(\mathrm{SO}_{n+1}) \rightarrow \pi_n(\mathrm{SO})$. By [20, pp. 522–525] this is precisely the freedom of choice we need to reduce $H_n(M;\mathbb{Z})$ to 2-torsion. We summarize what has been proved so far.

LEMMA 13. *There is a manifold W_1 obtained from $W_0 = M \times [0,1]$ by attaching $(n+1)$-handles to $M \times \{1\}$ such that W_1 admits an almost complex structure (so $\hat{\beta}_{W_1}$ is even) and $\partial W_1 = M \cup M_1$ with $H_n(M_1;\mathbb{Z}) = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$.*

Next we want to show that by attaching further $(n+1)$-handles to M_1 we can kill $H_n(M_1;\mathbb{Z})$ completely. Observe that W_1 is $(n-1)$-connected and trivially $\hat{\alpha}_{W_1}=0$ since the relevant coefficient group vanishes. From [19, Lemma 4.3] it follows, unless $n=7$, that $w_{n+1}(W_1)=0$. The proof of that same lemma shows that $w_{n+1}(W_1)$ is the image of $\hat{\beta}_{W_1}$ under a certain natural homomorphism, and since $\hat{\beta}_{W_1}$ is even we have $w_{n+1}(W_1)=0$ for $n=7$ as well.

The Wu formulae show $v_{n+1}=w_{n+1}=0$, and then

$$\begin{array}{ccc} \mathrm{Sq}^{n+1}: H^{n+1}(W_1,\partial W_1;\mathbb{Z}_2) & \to & H^{2n+1}(W_1,\partial W_1;\mathbb{Z}_2) \\ x & \mapsto & v_{n+1}\cup x \end{array}$$

is the zero map. Hence, by the argument on p. 525 of [20] we can attach further almost complex $(n+1)$-handles to kill $H_n(M_1;\mathbb{Z})$. We have proved the following

LEMMA 14. *There is an almost complex manifold W_2 obtained from $W_0=M\times[0,1]$ by attaching $(n+1)$-handles to $M\times\{1\}$ such that $\partial W_2=M\cup\Sigma^{2n+1}$ with Σ^{2n+1} a homotopy sphere.*

Now form the connected sum W_3 of the cobordism W_2 with the trivial cobordism $(-\Sigma^{2n+1})\times[0,1]$, where $-\Sigma^{2n+1}$ denotes the inverse element to Σ^{2n+1} in the group of homotopy spheres (which is simply Σ^{2n+1} with reversed orientation), and "connected sum" is understood in the sense of [20, Lemma 2.2]. That is, we choose a differentiable arc in W_2 joining a point in M with a point in Σ^{2n+1} such that a tubular neighbourhood of this arc is diffeomorphic to $\mathbb{R}^{2n+1}\times[0,1]$. Similarly, we remove an arc $*\times[0,1]$ from $(-\Sigma^{2n+1})\times[0,1]$, and then we glue the pointed tubular neighbourhoods of these arcs in the usual way.

The almost complex structure J_2 on W_2 induces an almost contact structure $(\xi,\mathscr{D},J_2\,|\,\mathscr{D})$ on Σ^{2n+1}, where $\mathscr{D}$ is the J_2-invariant subbundle of $T\Sigma^{2n+1}$ and the vector field ξ on Σ^{2n+1} is transverse to $\mathscr{D}$ and such that $J_2\xi$ is pointing inwards. This in turn induces natural almost complex structures $J_\pm$ on $(\pm\Sigma^{2n+1})\times[0,1]$, which differ only by the sign in $J_\pm\,\partial_t=\pm\,\xi$, where t denotes the coordinate in $[0,1]$.

There is an almost complex structure J_3 on W_3 which is uniquely determined up to homotopy by the condition that it extends the almost complex structures J_2 on W_2 and J_- on $(-\Sigma^{2n+1})\times[0,1]$), since over the tubular neighbourhoods of the two arcs along which we glue the almost complex structures define trivial $\mathrm{U}(n+1)$-bundles, and we identify the tangent bundles via an element in

$$[S^{2n}\times[0,1],\mathrm{SO}(2n+2)]\cong\pi_{2n}(\mathrm{SO}_{2n+2})\cong\pi_{2n}(\mathrm{SO})$$

which is the zero group for $n\equiv 7 \bmod 8$, and this lifts uniquely to an element of the zero group $\pi_{2n}(\mathrm{U}_{n+1})\cong\pi_{2n}(\mathrm{U})$.

The resulting almost contact structure on $\Sigma^{2n+1}\#(-\Sigma^{2n+1})$ at one end of W_3 is by our choice of almost contact structure on $-\Sigma^{2n+1}$ the one that extends over the contractible manifold B^{2n+2} bounded by $\Sigma^{2n+1}\#(-\Sigma^{2n+1})$ (see Fig. 1; notice that B^{2n+2} has $\Sigma^{2n+1}-D^{2n+1}$ as deformation retract).

In this figure the outer rim is identified with $\Sigma^{2n+1}=\Sigma^{2n+1}\times\{0\}$ and the inner rim with $-\Sigma^{2n+1}=\Sigma^{2n+1}\times\{1\}$, and the almost complex structure on $\Sigma^{2n+1}\times[0,1]$ is J_+. Then the almost complex structure J_+ on B^{2n+2} does indeed extend the almost complex structure J_3 on W_3. Thus we have found the desired almost complex handlebody

$$W=W_3\bigcup_{\Sigma^{2n+1}\#(-\Sigma^{2n+1})}B^{2n+2}$$

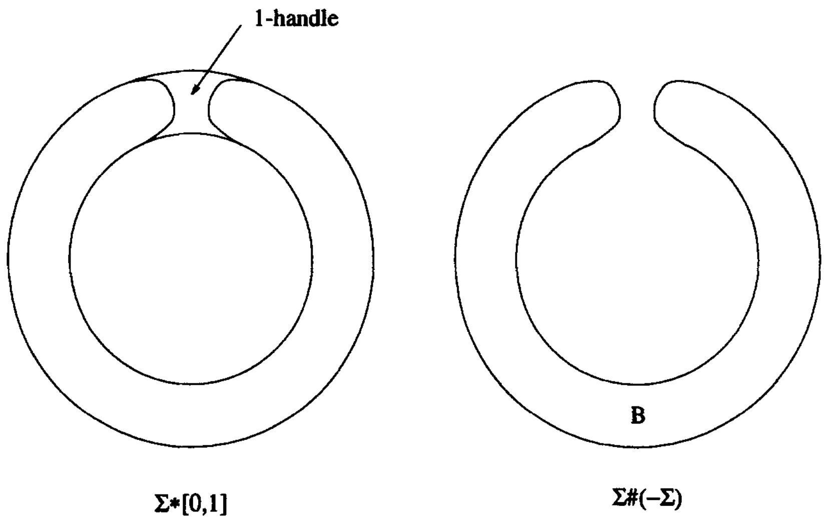

Fig. 1. The connected sum $\Sigma^{2n+1} \# (-\Sigma^{2n+1})$.

for the complementary handles we need to attach to a $(2n + 2)$-ball to obtain W are again of dimension $n + 1$. This concludes the proof of Theorem 4.

4.4. Some special cases

So far we have ruled out the case $\hat{\phi} \neq 0$ on the torsion-free part of $H_n(M; \mathbb{Z})$. In this section, we deal with this particular case.

PROPOSITION 15. *If* $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$, $\hat{\alpha}(M) = 0$, $\hat{\phi}(M) \neq 0$ *and M admits an almost contact structure, then M admits a contact structure.*

Proof. If $n \equiv 2, 4, 6 \bmod 8$, $\hat{\beta}$ lies in the zero group and so does its lift $\tilde{\beta}$, and there are no further middle dimensional obstructions to the existence of an almost contact structure. If $n \equiv 0 \bmod 8$, a necessary condition for M to admit an almost contact structure is $\hat{\beta} = 0$, and then the only lift is $\tilde{\beta} = 0$.

Hence the result is immediate from Lemma 11, since the cotangent sphere bundle of S^{n+1} (or any other manifold) is well-known to admit a contact structure.

Remark. (1) For $n \equiv 0, 4 \bmod 8$, the only lift of $\hat{\alpha} = 0$ is $\tilde{\alpha} = 0$. So in this case there is a unique stable homotopy class of almost contact structures. For $n \equiv 2 \bmod 8$, $\tilde{\alpha}$ can be any even number, for $n \equiv 6 \bmod 8$ any element of $\mathbb{Z}$. Since the cotangent disc bundle W in homotopy equivalent to S^{n+1}, only almost contact structures corresponding to $\tilde{\alpha} = 0$ can be induced from an almost complex structure on W.

(2) If $n \equiv 6 \bmod 8$, the condition $\hat{\alpha} = 0$ is trivially satisfied, since the relevant coefficient group is zero.

PROPOSITION 16. *If* $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$, $\hat{\alpha}(M) \neq 0$, $\hat{\phi} \neq 0$, *and M admits an almost contact structure, then $M \# M$ admits a contact structure.*

Proof. Let $e_1 = (1,0)$, $e_2 = (0,1)$ be generators of $\mathbb{Z} \oplus \mathbb{Z}$ with respect to the natural isomorphism $H_n(M\# M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Write e_1^*, e_2^* for the dual basis for $H^n(M\# M; \pi_{n-1}(\mathrm{SO})) \cong \mathrm{Hom}(H_n(M\# M;\mathbb{Z}), \pi_{n-1}(\mathrm{SO}))$. Then $\hat{\alpha}(M\# M) = \hat{\alpha}e_1^* + \hat{\alpha}e_2^*$, where $\hat{\alpha} = \hat{\alpha}(M)$.

The invariant $\hat{\phi}$ is an element of $H^{n+1}(M\# M;\mathbb{Z}_2)$. Since $H^{n+1}(M\# M)$ is Poincaré dual to $H_n(M\# M)$ and this duality is linear — in contrast to the anti-linear duality between $H_n(M\# M)$ and $\mathrm{Hom}(H_n(M\# M), \pi_{n-1}(\mathrm{SO}))$ — we can express $\hat{\phi}$ in terms of the basis e_1, e_2 reduced modulo 2. Thus $\hat{\phi}(M\# M) = e_1 + e_2$.

Now choose a new basis $\tilde{e}_1 = e_1$ and $\tilde{e}_2 = e_2 - e_1$. Then $\tilde{e}_1^* = e_1^* + e_2^*$ and $\tilde{e}_2^* = e_2^*$. Hence $\hat{\alpha}(M\# M) = \hat{\alpha}\tilde{e}_1^*$ and $\hat{\phi}(M\# M) = \tilde{e}_2$.

This proves that $M\# M = M'\# M''$, where $\hat{\alpha}(M') = \hat{\alpha}$, $\hat{\phi}(M') = 0$ and $\hat{\alpha}(M'') = 0$, $\hat{\phi}(M'') = 1$.

Because of $\hat{\phi}(M) \neq 0$ we are in the case n even (and $n \neq 2, 6$). Hence trivially $\hat{\beta}(M) = 0$ unless $n \equiv 0 \bmod 8$, in which case $\hat{\beta}(M) = 0$ is a necessary condition for M to admit an almost contact structure. Then also $\hat{\beta}(M') = \hat{\beta}(M'') = 0$.

The condition on $\hat{\alpha}(M')$ for M' to admit an almost contact structure is the same as for $\hat{\alpha}(M)$. Thus both M' and M'' (which in fact is the cotangent S^n-bundle of S^{n+1}) admit a contact structure by the discussion above, and hence so does $M\# M = M'\# M''$, obtained from M' and M'' by 0-surgery.

5. EXOTIC CONTACT STRUCTURES ON SPHERES

Non-standard contact structures on spheres of dimension $\geqslant 5$ have been constructed by Eliashberg [2]. On S^3 exotic contact structures were discovered by Bennequin and completely classified by Eliashberg [3].

In dimension $\geqslant 5$ the construction rests on a theorem of Gromov, Eliashberg, Floer and McDuff which states that if (W, Ω) is a symplectic manifold with contact type boundary equal to S^{2n+1} with its standard contact structure and (W, Ω) does not contain any symplectic 2-spheres, then W is diffeomorphic to a ball B^{2n+2}. Eliashberg uses a plumbing construction to obtain contact structures on S^{2n+1}, $n \geqslant 2$, with a symplectic filling different from B^{2n+2}, which proves exoticity of these structures.

Homotopy classes of almost contact structures on S^{2n+1} are classified by $\pi_{2n+1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n)$. Recall Massey's computation of these homotopy groups from Section 4, which shows that there are finitely many such homotopy classes if n is even and infinitely many if n is odd. In the former case, Eliashberg could actually produce exotic contact structures on S^{2n+1} that are homotopically standard, simply by taking the connected sum of k copies of an arbitrary exotic example, where k is divisible by the order of $\pi_{2n+1}(\mathrm{SO}_{2n+1}/\mathrm{U}_n)$. For n odd this simple procedure fails, and examples of homotopically non-standard contact structures have been previously known to Sato [29].

The purpose of the present section is two-fold. First we outline an alternative construction of exotic contact structures on spheres by describing spheres as Brieskorn manifolds. This description — which is essentially Sato's, except that he was only interested in the homotopy problem — is of course related to Eliashberg's construction because of the well-known relation between Brieskorn manifolds and plumbing constructions [17]. However, it has the advantage not only of giving an explicit global description of the contact structure, but it also allows to determine the homotopy class of the underlying almost contact structure.

While this construction is not directly related to contact surgery, it prepares the ground for the second part of this section. We show that although the construction via Brieskorn manifolds never leads to exotic contact structures on S^7 that are homotopically standard, such examples can indeed be constructed using contact surgery instead. We also construct such examples on S^{8k+3}, $k \geqslant 1$. Some of the results in this section were announced in [10].

5.1. Brieskorn manifolds

Consider the Brieskorn manifold $\Sigma(a) = \Sigma(a_0, \ldots, a_{n+1})$, defined as intersection of the non-singular complex hypersurface given by the equation

$$z_0^{a_0} + \cdots + z_{n+1}^{a_{n+1}} = t$$

in $\mathbb{C}^{n+2}$ (with t a small positive real number) with the unit sphere $S^{2n+3} \subset \mathbb{C}^{n+2}$. Here the a_i are natural numbers, $a_i \geqslant 2$. From [21] and the discussion in [17, Chapter 14] it follows that the real 1-form

$$\omega = \frac{i}{2} \sum_{j=0}^{n+1} \frac{1}{a_j} (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

defines (for sufficiently small t) a contact form on $\Sigma(a)$. For suitable choice of $a = (a_0, \ldots, a_{n+1})$ the Brieskorn manifold $\Sigma(a)$ is diffeomorphic to S^{2n+1} and it admits a symplectic (in fact, holomorphic) filling $(W(a), \Omega)$ given by

$$W(a) = \left\{ (z_0, \ldots, z_{n+1}) \in \mathbb{C}^{n+2} \,\middle|\, \sum_{j=0}^{n+1} z_j^{a_j} = t, \ \sum_{j=0}^{n+1} |z_j|^2 \leqslant 1 \right\}$$

$$\Omega = i \sum_{j=0}^{n+1} \frac{1}{a_j} dz_j \wedge d\bar{z}_j.$$

The homology of $W(a)$ can be computed explicitly and this allows to show the exoticity of the contact structure induced by ω on $\Sigma(a)$.

PROPOSITION 17. *Suppose $\Sigma(a)$ is diffeomorphic to S^{2n+1}, $n \geqslant 2$, $a \neq (2, \ldots, 2)$. Then the contact structure induced by ω is exotic.*

Proof. As shown in [17], the cobounding manifold $W(a)$ is an n-connected $(2n+2)$-manifold with

$$\operatorname{rank} H_{n+1}(W(a)) = \prod_{i=1}^{n+1} (a_i - 1) \neq 0.$$

In particular, $W(a)$ is not a ball. Now apply the theorem from [2] mentioned at the beginning of this section.

Notation. We write $\mu(a) = \prod_{i=1}^{n+1} (a_i - 1)$ and sign(a) for the signature of $W(a)$.

EXAMPLE 18. $\Sigma(2, 2, 2, 3, 6{\cdot}28 - 1)$ *is diffeomorphic to S^7 and the contact structure induced by ω is exotic.*

Proof. This follows from [17, Satz 14.7] and the proposition above.

This explicit description has the further advantage that the homotopy class of the underlying almost contact structure can be determined. In fact, one can show that for S^7 such a construction always yields a contact structure whose underlying almost contact structure is not homotopic to the standard structure (and probably the same is true for any S^{2n+1} with n odd).

PROPOSITION 19. *Suppose $\Sigma(a) = \Sigma(a_0, \dots, a_4)$ is diffeomorphic to S^7. The contact structure induced by ω is not homotopically standard.*

Proof. Morita [27] gives an explicit calculation of the homotopy class of the induced almost contact structure. Recall that these homotopy classes are classified by $\pi_7(SO_7/U_3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, where $(0,0)$ corresponds to the standard structure (which extends over D^8 as an almost complex structure). Write $\delta(a) \in \pi_7(SO_7/U_3)$ for the homotopy class induced on $\Sigma(a)$. By Proposition 3.1 of [27] we have

$$\delta(a) = \left(\frac{9}{28}\operatorname{sign}(a) - \frac{1}{2}\mu(a), 0\right) \in \mathbb{Z} \oplus \mathbb{Z}_2.$$

Write a^+ (resp. a^-) for the dimension of the subspace of $H_{n+1}(W(a))$ on which the intersection form of $W(a)$ is positive (resp. negative) definite. Then $\operatorname{sign}(a) = a^+ - a^-$ and $\mu(a) = a^+ + a^-$. It follows

$$\frac{9}{28}\operatorname{sign}(a) - \frac{1}{2}\mu(a) \leqslant \frac{9}{28}a^+ - \frac{1}{2}a^+ \leqslant 0$$

where at least one of the inequalities is strict.

Remark. (1) Here the splitting $\pi_7(SO_7/U_3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ is chosen such that $1 \in \mathbb{Z}$ generates the kernel of the stabilizing map $\pi_7(SO_7/U_3) \to \pi_7(SO/U)$, $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$; cf. Section 5.2.

(2) Notice that even stably we can only realize the class $y = 0 \in \mathbb{Z}_2$. It is not known whether there are contact structures on S^7 in the homotopy classes of almost contact structures of the form $(x, y) \in \mathbb{Z} \oplus \mathbb{Z}_2$ with $y \neq 0$.

For n even, one can give explicit realizations of contact structures on S^{2n+1} in every homotopy class of almost contact structures. The following theorem is essentially due to Morita [27], rephrased in the language of contact geometry, cf. [29]. However, our proof differs slightly from Morita's and uses contact surgery. We alluded to this theorem in Remark (2) after Theorem 4.

THEOREM 20. *On S^{2n+1}, n even, every homotopy class of almost contact structures contains a contact structure.*

Proof. Consider

$$\Sigma(a) = \Sigma(\overbrace{2, \dots, 2}^{n+1}, 2k+1)$$

with $2k + 1 \equiv \pm 1 \bmod 8$. By [17, Satz 11.3], $\Sigma(a)$ is diffeomorphic to S^{2n+1}. Morita computes the homotopy class of the almost contact structure induced from ω; he finds $\delta(a) = (1/2)\mu(a) = k \in G := \pi_{2n+1}(SO_{2n+1}/U_n)$, which is a finite group. For $n = 2$ this group is trivial, so assume $n \geqslant 4$. Then the order $|G|$ of this group is divisible by four, so for $k = |G| - 1$ we have $2k + 1 \equiv -1 \bmod 8$. Taking connected sums of the corresponding $\Sigma(a)$ produces a contact structure in any homotopy class of almost contact structures, for the δ-invariant is easily seen to be additive under connected sums (cf. Section 5.2).

One can also give concrete examples where the contact structure on S^{2n+1} is exotic but homotopically standard.

EXAMPLE 21. *$\Sigma(2,2,2,2,2,49)$ is diffeomorphic to S^9 and the induced contact structure is exotic but homotopically standard.*

Proof. The order of $\pi_9(\mathrm{SO}_9/\mathrm{U}_4)$ is 24.

5.2. Almost X-structures

In this paragraph we briefly review work of Kahn [18] and relate it to the homotopy questions studied in the preceding paragraph.

An *X-structure* as defined by Kahn on an oriented manifold W is a section σ of the bundle associated to the tangent bundle TW with a suitable fibre X. Suffice it to say that the concept includes the case of almost complex structures ($X = \mathrm{SO}_{2n+2}/U_{n+1}$ for $\dim W = 2n+2$) and other reductions of the structure group of W.

An *almost X-structure* in the sense of Kahn is an X-structure on W with a disc removed. (Note the two different meanings of *almost*). Given an almost X-structure σ on W, there is an obstruction $c_X(W,\sigma)$ to extending this to an X-structure. This obstruction only depends on the homotopy class of the almost X-structure σ.

Given almost X-structures σ_1, σ_2 on two manifolds V_1, V_2 of the same dimension m, one can form the connected sum $V_1 \# V_2$ with respect to the discs $D_i^m \subset V_i$ outside which the X-structures are defined and extend the given X-structures over the 1-handle with a disc removed. This almost X-structure on $V_1 \# V_2$ is denoted by $\sigma_1 + \sigma_2$. Similarly, an almost X-structure σ on V gives rise to an almost X-structure $-\sigma$ on $-V$, that is, V with orientation reversed.

Let $c_X(S^m)$ denote the obstruction to extending any almost X-structure on S^m over the whole sphere, and denote by $\chi(V)$ the Euler characteristic of V. Then we have the following theorem of Kahn, which extends the well-known formulae for the Euler characteristic, interpreted in this context as the obstruction to the existence of a nowhere vanishing vector field.

THEOREM 22 (Kahn [18]).

(i) $c_X(V_1^m \# V_2^m, \sigma_1 + \sigma_2) = c_X(V_1^m, \sigma_1) + c_X(V_2^m, \sigma_2) - c_X(S^m)$,
(ii) $c_X(-V^m, -\sigma) = -c_X(V^m, \sigma) + \chi(V) c_X(S^m)$.

We have an immediate application for the case $m = 2n+1$, $X = \mathrm{SO}_{2n+1}/\mathrm{U}_n$, that is, almost contact structures. This was used in Section 4.

PROPOSITION 23. (i) *The top dimensional obstruction to the existence of an almost contact structure is additive under the connected sum of manifolds. In particular, the connected sum of two almost contact manifolds admits an almost contact structure.*

(ii) *A manifold that is almost diffeomorphic to an almost contact manifold also admits an almost contact structure.*

Proof. These statements follow immediately from the fact that every homotopy sphere is stably parallelizable [20], so the odd-dimensional ones admit an almost contact structure.

We now specialize to the case of almost complex structures in dimension $2n+2$, n odd, so we set $X = \mathrm{SO}_{2n+2}/\mathrm{U}_{n+1}$. Notice that $\mathrm{SO}_{2n+2}/\mathrm{U}_{n+1}$ is diffeomorphic to SO_{2n+1}/U_n by [12], in particular the obstruction c_X to extending an almost X-structure lies in

$$\pi_{2n+1}(SO_{2n+2}/U_{n+1}) \cong \pi_{2n+1}(SO_{2n+1}/U_n) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 1 \bmod 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n \equiv 3 \bmod 4 \end{cases}$$

which is also the group classifying homotopy classes of almost contact structures on S^{2n+1}.

First we want to compute $c_X(S^{2n+2}) \in \pi_{2n+1}(SO_{2n+2}/U_{n+1})$. Observe that, for $n \equiv 3 \bmod 4$, the stabilizing map

$$\begin{array}{ccc} S\colon \pi_{2n+1}(SO_{2n+2}/U_{n+1}) & \to & \pi_{2n+1}(SO/U) \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \to & \mathbb{Z}_2 \end{array}$$

is surjective, since

$$\begin{array}{ccccc} \pi_{2n+1}(SO_{2n+2}) & \to & \pi_{2n+1}(SO) & \to & \pi_{2n+1}(SO/U) \\ \mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}_2 \end{array}$$

is surjective. Define the splitting $\pi_{2n+1}(SO_{2n+2}/U_{n+1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ by identifying $\ker S$ with $\mathbb{Z}$.

From [23, Theorem II] it follows that, for $n \equiv 1, 3 \bmod 4$, $c_X(S^{2n+2})$ generates $\pi_{2n+1}(SO_{2n+2}/U_{n+1})$ (resp. its free part), so from the fact that S^{2n+2} is stably parallelizable we have (after choosing a sign)

$$c_X(S^{2n+2}) = \begin{cases} 1 \in \mathbb{Z} & \text{for } n \equiv 1 \bmod 4 \\ (1,0) \in \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n \equiv 3 \bmod 4. \end{cases}$$

Observe the following relation with Morita's invariant δ discussed in the preceding section. Let W be a closed $(2n+2)$-manifold such that $W - U^{2n+2}$ (where U^{2n+2} denotes the interior of an embedded closed $(2n+2)$-disc $D^{2n+2} \subset W$) admits an almost complex structure. The induced almost contact structure on $\partial(W - U^{2n+2}) = S^{2n+1}$ determines an element $\delta \in \pi_{2n+1}(SO_{2n+1}/U_n)$ with $\delta = 0$ precisely if S^{2n+1} bounds an almost complex $(2n+2)$-ball. In this way we may regard δ as an invariant associated with W and an almost complex structure σ on $W - U^{2n+2}$. Our convention implies $\delta(S^{2n+2}) = 0$. We claim that the relation

$$\delta(W,\sigma) + c_X(W,\sigma) = c_X(S^{2n+2}) = (1,0)$$

holds for any $(2n+2)$-manifold W. Indeed, consider an almost contact structure induced on $S^{2n+1} = \partial(W - U^{2n+2})$ from an almost complex structure σ on W. From the orientation of $W - U^{2n+2}$ given by the almost complex structure, S^{2n+1} inherits a natural orientation as boundary. The invariant $\delta(W,\sigma)$ may be regarded as the obstruction to extending the almost contact structure on S^{2n+1} to an almost complex structure on an $(2n+2)$-ball inside S^{2n+1}, the class $c_X(W,\sigma)$ as the obstruction to extending it to an $(2n+2)$-ball outside S^{2n+1}. So these two obstructions must add up to the obstruction to finding an almost complex structure on S^{2n+2}.

Observe further that δ is additive under connected sums. This follows from the fact that the Eliashberg–Weinstein 0-surgery yields an almost contact structure on the connected sum that extends as a complex structure over the 1-handle. In the present situation, this is equivalent to Kahn's theorem:

$$\begin{aligned} \delta(W_1 \# W_2, \sigma_1 + \sigma_2) &= (1,0) - c_X(W_1 \# W_2, \sigma_1 + \sigma_2) \\ &= (1,0) - (c_X(W_1,\sigma_1) + c_X(W_2,\sigma_2) - (1,0)) \\ &= ((1,0) - (c_X(W_1,\sigma_1)) + ((1,0) - c_X(W_2,\sigma_2)) \\ &= \delta(W_1,\sigma_1) + \delta(W_2,\sigma_2). \end{aligned}$$

5.3. Exotic structures on S^{2n+1}, *n odd*

We now apply the results of the preceding paragraph to construct exotic but homotopically standard contact structures on S^7 and S^{8k+3}, $k \geqslant 1$. The construction, which at least in principle extends to all spheres of dimension $2n+1$ with n odd, rests on the following theorem.

THEOREM 24. *Let W_0 be an oriented n-connected (but not $(n+1)$-connected) $(2n+2)$-manifold that admits an almost complex structure J_0 (which defines the orientation), where n is odd. Observe that $\chi(W_0) > 0$. Set*

$$\bar{W} = \underbrace{W_0 \# \cdots \# W_0}_{\chi(W_0)-1} \# (-W_0).$$

Then $W = \bar{W} - U^{2n+2}$ admits a symplectic structure and a compatible almost complex structure which induce an exotic but homotopically standard contact structure on $\partial W = S^{2n+1}$.

Proof. Set $X = \mathrm{SO}_{2n+2}/\mathrm{U}_{n+1}$. Denote by σ the almost X-structure on W_0 induced by J_0. Clearly $c_X(W_0, \sigma) = 0$. Now we compute

$$\begin{aligned} c_X(-W_0, -\sigma) &= -c_X(W_0,\sigma) + \chi(W_0)c_X(S^{2n+2}) \\ &= \chi(W_0)c_X(S^{2n+2}) \end{aligned}$$

and further

$$\begin{aligned} c_X(\bar{W}, (\chi(W_0)-1)\sigma + (-\sigma)) &= (\chi(W_0)-1)c_X(W_0,\sigma) + c_X(-W_0, -\sigma) \\ &\quad - (\chi(W_0)-1)c_X(S^{2n+2}) \\ &= c_X(S^{2n+2}). \end{aligned}$$

It follows that W admits an almost complex structure whose δ-invariant is $0 \in \mathbb{Z}$ (resp. $(0,0) \in \mathbb{Z} \oplus \mathbb{Z}_2$).

By Smale's classical work on the generalized Poincaré conjecture it is well known that W is a handlebody in $\mathscr{H}(2n+2, k, n+1)$, so we may apply Theorem 3 to conclude that $\partial W = S^{2n+1}$ admits a homotopically standard contact structure, which is exotic because of $H_{n+1}(W) \neq 0$

Next we are going to construct the highly connected almost complex manifolds $\bar{W}$ required by the theorem we have just proved.

THEOREM 25. (i) *$\mathbb{H}P^2 \# \mathbb{H}P^2 \# S^4 \times S^4$ admits an almost complex structure.*

(ii) *Fix an orientation on $V = S^{4k+2} \times S^{4k+2}$, $k \geqslant 1$. There is a connected sum of copies of V and $-V$ which admits an almost complex structure.*

(iii) *A connected sum of copies of $S^{4k} \times S^{4k}$ with either orientation does not admit an almost complex structure.*

COROLLARY 26. *S^7 and S^{8k+3}, $k \geqslant 1$, admit exotic but homotopically standard contact structures.*

This corollary is immediate from the two preceding theorems. Part (iii) of the theorem above shows that the case of spheres of dimension $8k-1$ with $k \geqslant 2$ is more complicated. Similar considerations (using Proposition 3.1 of [27]) show that for $k = 1$, and probably also for all $k \geqslant 2$ — this depends on having suitable estimates on the Bernoulli numbers — the boundary connected sum of manifolds $W(a)$ as in Section 5.1 with $\partial W(a)$ a homotopy sphere never leads to a homotopically standard contact structure on S^{8k-1}. In particular, we leave open the question whether there are any highly connected almost complex manifolds in dimension $8k$, $k \geqslant 2$. The case $k = 1$ is exceptional because we can use $\mathbb{H}P^2$ as building block.

Proof of Theorem 25. (i) First we consider $\mathbb{H}P^2$. By [15] the total Pontrjagin class of the tangent bundle $\theta = T\mathbb{H}P^2$ is

$$p(\theta) = (1+u)^6(1+4u)^{-1} = 1 + 2u + 7u^2$$

where u is a suitable generator of $H^4(\mathbb{H}P^2;\mathbb{Z}) \cong \mathbb{Z}$. Since $\pi_3(\mathrm{SO}_8/\mathrm{U}_4) \cong \pi_3(\mathrm{SO}/\mathrm{U}) = 0$, the structure group of θ reduces to U_4 over the 4-skeleton $S^4 = \mathbb{H}P^1 \subset \mathbb{H}P^2$. Write η for the resulting U_4-bundle over S^4 and σ for the corresponding almost X-structure on $H\mathbb{P}^2$ (with $X = \mathrm{SO}_8/\mathrm{U}_4$). The relation

$$p_1(\theta) = c_1^2(\eta) - 2c_2(\eta)$$

implies $c_2(\eta) = -u$. By [23, Theorem II] or [18, Corollary 2] we find for the free summand c_X^0 of $c_X = c_X^0 + c_X^2 \in \mathbb{Z} \oplus \mathbb{Z}_2$ that

$$\begin{aligned} c_X^0(\mathbb{H}P^2,\sigma) &= \tfrac{1}{2}(\chi(\mathbb{H}P^2) + \tfrac{1}{2}\langle c_2^2(\eta) - p_2(\theta), [\mathbb{H}P^2]\rangle)\,c_X^0(S^8) \\ &= \tfrac{1}{2}(3 - 3\langle u^2, [\mathbb{H}P^2]\rangle)c_X^0(S^8) \end{aligned}$$

where $[\mathbb{H}P^2]$ denotes the orientation generator of $H_8(\mathbb{H}P^2;\mathbb{Z})$. Now u^2 is a generator of $H^8(\mathbb{H}P^2;\mathbb{Z})$, so if we define the orientation of $\mathbb{H}P^2$ by the condition $\langle u^2, [\mathbb{H}P^2]\rangle = 1$, then $c_X^0(\mathbb{H}P^2,\sigma) = 0$. However, $\mathbb{H}P^2$ (with either orientation) does not admit any almost complex structures, since a necessary condition for an 8-manifold with vanishing second Betti number to do so is that its Euler number be divisible by 6, see [16]. We conclude

$$c_X(\mathbb{H}P^2,\sigma) = (0,1) \in \mathbb{Z} \oplus \mathbb{Z}_2.$$

Next we compute c_X for $S^4 \times S^4$. Again we can find an almost complex structure over the 4-skeleton $S^4 \vee S^4$, and hence an almost X-structure σ' on $S^4 \times S^4$. This manifold is stably parallelizable, so its total Pontrjagin class is equal to 1. It follows that $c_2(\eta') = 0$, where η' denotes the U_4-bundle over $S^4 \vee S^4$ given by σ'. Thus we find

$$c_X^0(S^4 \times S^4, \sigma') = \tfrac{1}{2}\chi(S^4 \times S^4)c_X^0(S^8) = 2c_X^0(S^8)$$

and hence, because $S^4 \times S^4$ is stably almost complex (being stably parallelizable),

$$c_X(S^4 \times S^4, \sigma') = (2,0) \in \mathbb{Z} \oplus \mathbb{Z}_2.$$

Now we compute

$$\begin{aligned} c_X(\mathbb{H}P^2 \# \mathbb{H}P^2 \# S^4 \times S^4, \sigma + \sigma + \sigma') &= (0,1) + (0,1) + (2,0) - 2(1,0) \\ &= (0,0) \end{aligned}$$

so $\bar{W} = \mathbb{H}P^2 \# \mathbb{H}P^2 \# S^4 \times S^4$ admits an almost complex structure. Notice that $\chi(\bar{W}) = 3 + 3 + 4 - 2 \times 2 = 6$, so the criterion of [16] is satisfied.

(ii) First of all we observe again that $S^{4k+2} \times S^{4k+2}$ is stably parallelizable, so its total Pontrjagin class equals 1. Since $\pi_{4k+1}(\mathrm{SO}_{8k+4}/\mathrm{U}_{4k+2}) \cong \pi_{4k+1}(\mathrm{SO}/\mathrm{U}) = 0$, we can find an almost complex structure σ over the $(4k+2)$-skeleton $S^{4k+2} \vee S^{4k+2}$. By [19, Lemma 1.1] the Chern class $c_{2k+1}(\eta)$ of the corresponding $\mathrm{U}(4k+2)$-bundle η can be any multiple of

$(2k)!$ (The mod 2 reduction of this Chern class is equal to zero, so the underlying real bundle is always the trivial one). Choose generators x, y of

$$H^{4k+2}(S^{4k+2} \vee S^{4k+2}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

and η with

$$c_{2k+1}(\eta) = (2k)!x + (2k)!y.$$

Write V for $S^{4k+2} \times S^{4k+2}$ with the orientation defined by $x \cup y$. Then, by the results of Massey and Kahn cited above,

$$\begin{aligned} c_X(V, \sigma) &= (\tfrac{1}{2}\chi(V) - \tfrac{1}{4}2(2k)!(2k)!)c_X(S^{8k+4}) \\ &= (2 - \tfrac{1}{2}(2k)!(2k)!)c_X(S^{8k+4}) \\ &=: (2 - a_k)c_X(S^{8k+4}) \in \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} c_X(-V, -\sigma) &= (-(2 - a_k) + \chi(V))c_X(S^{8k+4}) \\ &= (2 + a_k)c_X(S^{8k+4}). \end{aligned}$$

Recall that $c_X(S^{8k+4})$ is a generator of $\mathbb{Z} \cong \pi_{8k+3}(\mathrm{SO}_{8k+4}/\mathrm{U}_{4k+2})$.

Now consider

$$\bar{W} = \#_a V \# (\#_b(-V))$$

and

$$\bar{\sigma} = \bigoplus_a \sigma \oplus \left(\bigoplus_b (-\sigma)\right)$$

with $a = a_k/2$ and $b = a_k/2 - 1$. Then

$$\begin{aligned} c_X(\bar{W}, \bar{\sigma}) &= a(2 - a_k) + b(2 + a_k) - a - b + 1 \\ &= \frac{a_k}{2}(1 - a_k) + \left(\frac{a_k}{2} - 1\right)(1 + a_k) + 1 \\ &= 0, \end{aligned}$$

so $\bar{W}$ admits an almost complex structure. Observe that for $k = 1$ we obtain $\bar{W} = S^6 \times S^6$. Indeed, S^6 admits an almost complex structure.

(iii) Suppose we have an almost complex structure over the $4k$-skeleton $S^{4k} \vee S^{4k}$ of $S^{4k} \times S^{4k}$. With notation as above, we have

$$0 = p_k(\theta) = \pm c_{2k}(\eta)$$

and hence

$$c_X(S^{4k} \times S^{4k}, \sigma) = 2c_X(S^{4k})$$

independently of the orientation, as can be seen by passing to $-\sigma$. Therefore no connected sum of copies of $S^{4k} \times S^{4k}$ with either orientation has vanishing obstruction c_X.

6. NON-LINEAR SPHERICAL SPACE FORMS

A *spherical space form* is a quotient manifold $M = \Gamma \backslash S^k$ with Γ a finite group acting freely on S^k. If M is diffeomorphic to a quotient $\Gamma \backslash S^k$ with $\Gamma \subset O(k + 1)$ it is called a *linear* spherical space form. Wolf [37] has shown the existence of a contact structure on any odd-dimensional linear spherical space form. In this section we employ contact surgery to produce contact structures on certain non-linear spherical space forms. We limit ourselves to the case $k = 5$, although some of the arguments can easily be generalized.

Consider the Brieskorn 5-manifold $\Sigma_{p,l} = \Sigma_{p,l}(\varepsilon, \eta) \subset \mathbb{C}^4$ defined by the equations

$$z_0^p + z_1^p + z_2^p + z_3^l = \varepsilon$$

$$\sum_{j=0}^{3} |z_j|^2 = \eta.$$

Here ε and η are positive real numbers, p is an odd integer, and $l = 3^n$ for some natural number n. Furthermore, assume that r is a primitive third root of $1 \bmod p$ and $\gcd(r - 1, p) = 1$. It is shown in [28] that under these assumptions the metacyclic group

$$D_{p,3} = \{x, y \mid x^p = y^3 = 1,\ yxy^{-1} = x^r\}$$

acts freely on $\Sigma_{p,l}$ for appropriate choice of ε and η (for instance, it suffices to require $\eta^l \neq \varepsilon$ and $\eta^p 3^{1-p/2} \neq \varepsilon$). An example is given by $p = 7$, $r = 2$.

This action of $D_{p,3}$ on $\Sigma_{p,l}$ is given by

$$x(z_0, z_1, z_2, z_3) = (\xi_p z_0, \xi_p^r z_1, \xi_p^{r^2} z_2, z_3)$$

and

$$y(z_0, z_1, z_2, z_3) = (z_2, z_0, z_1, \xi_3 z_3)$$

where ξ_p, ξ_3 denote a primitive pth and 3rd root of unity, respectively.

As in Section 5 we know that

$$\omega = \frac{i}{2}\left(\frac{1}{p}\sum_{j=0}^{2} (z_j\, d\bar{z}_j - \bar{z}_j\, dz_j) + \frac{1}{l}(z_3\, d\bar{z}_3 - \bar{z}_3\, dz_3)\right)$$

induces a contact form on $\Sigma_{p,l}$ for ε sufficiently small.

We observe that ω is invariant under the action of $D_{p,3}$ and hence descends to a contact form on the quotient manifold $D_{p,3}\backslash\Sigma_{p,l}$. The manifold $\Sigma_{p,l}$ is not a 5-sphere, but it is simply connected and $H_2(\Sigma_{p,l}; \mathbb{Z})$ is a finite group.

Petrie [28] has shown that for suitable choice of n it is possible to obtain a manifold with fundamental group $D_{p,3}$ and universal cover S^5 by performing surgery along 2-spheres in $Q_{p,l} := D_{p,3}\backslash\Sigma_{p,l}$. In other words, this constructs an action of $D_{p,3}$ on S^5. This action is necessarily non-linear since the group-theoretic conditions of [38, Theorem 5.5.1] for a linear action are violated.

Our aim now is to show that this surgery can be carried out so as to preserve the contact property. The idea that Petrie's construction ought to carry over to contact geometry belongs to Charles Thomas.

THEOREM 27. *There is a contact structure on $D_{p,3}\backslash S^5$.*

Proof. First we are going to show that the 2-spheres along which we want to perform surgery satisfy the conditions of Lemma 1. By Remark (2) after Lemma 2, it is then possible to perform contact surgery. Now $H_2(\Sigma_{p,l}; \mathbb{Z})$ is finite and so is $H_2(Q_{p,l}; \mathbb{Z})$ (One can compute directly that $H_2(D_{p,3})$ is finite and then the finiteness of $H_2(Q_{p,l}; \mathbb{Z})$ follows from the Cartan–Leray spectral sequence, which has E_2-page given by

$$E_2^{s,t} \cong H_s(D_{p,3}; H_t(\Sigma_{p,l}; \mathbb{Z}))$$

and converges to the homology of $Q_{p,l}$). Hence (with $\mathbb{Z}$-coefficients understood)

$$\begin{aligned} H^2(Q_{p,l}) &\cong FH_2(Q_{p,l}) \oplus TH_1(Q_{p,l}) \\ &\cong T(D_{p,3})_{ab} \cong \mathbb{Z}_3. \end{aligned}$$

Here F denotes the free part of an abelian group, T the torsion part, and $(.)_{ab}$ the abelianization of a group.

Now let $i: S^2 \to Q_{p,l}$ be an embedding and $\mathscr{D}$ the contact structure on $Q_{p,l}$. Since $i^*: H^2(Q_{p,l}; \mathbb{Z}) \to H^2(S^2; \mathbb{Z})$ must be the trivial homomorphism, both $TS^2 \otimes \mathbb{C}$ and $i^*\mathscr{D}$ are U(2)-bundles over S^2 with $c_1 = 0$ and therefore trivial bundles. Hence we can find the desired complex bundle isomorphism $TS^2 \otimes \mathbb{C} \to \mathscr{D}|i(S^2)$.

Thus it is possible to perform contact surgery along all the relevant 2-spheres. In general, there is a choice of framing and not every framing may be realizable by a contact surgery. However, in the present situation the choice of framing lies in $\pi_2(SO_3) = 0$, so this problem does not arise.

In forthcoming joint work with C. B. Thomas we intend to study this and related constructions in more detail.

7. CONVEX SYMPLECTIC MANIFOLDS

The manifolds constructed in Section 4 are actually all Ω-convex boundaries of suitable symplectic handlebodies (W, Ω). Indeed they are even holomorphically fillable in the sense of [4]. Notice that in all cases W^{2n+2} has the homotopy type of an $(n+1)$-complex.

In [4] Eliashberg and Gromov asked whether there exist complete (globally) convex symplectic manifolds of dimension $2n \geqslant 6$ with non-trivial $(2n-1)$-dimensional homology. The relevance of this question is discussed in great detail in their paper. For $2n = 4$ such examples had first been constructed by McDuff [24], and a generalization of her construction for $2n = 4$ was given independently in [9, 26], and for $2n = 6$ in [8], thus answering the above question in the affirmative.

Strictly speaking, in [8] it was only shown that there are examples which show that there are *locally* convex symplectic manifolds with the described homological property. We briefly recall the construction from [8] to show that it actually yields *globally* convex symplectic manifolds (which can easily be made complete by the completion procedure of [4, Section 1.8.4]). In [8] it was shown that on the total space M of suitable T^3-bundles over T^2 one can find 1-forms $\alpha_1, \dots, \alpha_5$ satisfying

$$\begin{aligned} d\alpha_1 &= \alpha_4 \wedge \alpha_2 + \alpha_5 \wedge \alpha_3 \\ d\alpha_2 &= \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_1. \end{aligned}$$

Let $\psi: [0,1] \to [0,1]$ be a smooth, monotone increasing function with $\psi(s)$ identically zero near $s = 0$ and $\psi(s) = s$ near $s = 1$ and such that for each $s \in [0,1]$ at least one of $\psi'(s)\psi(s)$ and $\psi'(1-s)\psi(1-s)$ is strictly positive. Set

$$\Omega = d(\psi(s)\alpha_1 + \psi(1-s)\alpha_2)$$

on $W = M \times [0,1]$. Then one easily computes that Ω is a symplectic form on W. Furthermore, the vector field X globally defined on W by

$$\Omega(X, .) = \psi(s)\alpha_1 + \psi(1-s)\alpha_2$$

is clearly an expanding Liouville vector field; near $s = 0$ we find $X = -(1-s)\partial_s$ and near $s = 1$ we have $X = s\partial_s$, hence X is pointing outwards along the boundary. Thus W is a convex symplectic 6-manifold with $H_5(W;\mathbb{Z}) \cong \mathbb{Z}$.

The earlier 4-dimensional examples in [22, 9, 26] were also of the special form $M^3 \times [0,1]$, but using contact surgery one can readily obtain convex symplectic manifolds with more intricate topology. By attaching symplectic handles to one of the boundary components of $M \times [0,1]$ one can construct convex symplectic manifolds with non-diffeomorphic boundary components, and by repeated attaching of 1-handles between convex symplectic manifolds (i.e. forming of the boundary connected sum) one obtains symplectic manifolds with arbitrary number of boundary components.

Acknowledgements—I am deeply grateful to Ya. Eliashberg for helpful conversations and explanations concerning his work on contact surgery, and to C. B. Thomas for sharing his ideas with me.

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